

# Introduction to the Classical Theory of Higher Spins\*

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**ABSTRACT:** We review main features and problems of higher spin field theory and flash some ways along which it has been developed over last decades.

**KEYWORDS:** string theory, field theory, higher spin fields.

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## 1. Introduction

Several dozens of years of intensive study, which involved enormous amount of theoretical brain power, have resulted in a deep insight into various fundamental features of String Theory, and every new year of research brings us new and new aspects of its immense structure. By now, for example, we know in detail low energy field-theoretical limits of String Theory which correspond to massless excitations over different string vacua. These are described by ten-dimensional supergravities whose supermultiplets consist of fields of

spin not higher than two and higher order corrections thereof. This region of String Theory also contains various types of branes which reflect dualities between different string vacua. The classical dynamics of the supergravity fields is well known from the analysis of the classical supergravity actions and equations of motion which possess an interesting geometrical and symmetry structure based on supersymmetry. However, from the perspective of quantization the ten-dimensional supergravities look not so promising since they are non-renormalizable as field theories containing gravitation usually are. At the same time (Super)String Theory is believed to be a renormalizable and even finite quantum theory in the ultraviolet limit, and therefore it consistently describes quantum gravity. A field theoretical reason behind this consistent quantum behavior is the contribution to quantum corrections of an infinite tower of massive higher spin excitations of the string, whose mass squared is proportional to string tension and spin (e.g. in open string theory  $M_s^2 \sim T(s-1) \sim \frac{1}{\alpha'}(s-1)$ , where  $s$  is the maximum spin value of a state). Therefore, a better understanding of the dynamics of higher spin states is important for the analysis of quantum properties of String Theory.

Until recently the field of higher spins has remained a virgin land cultivated by only a few enthusiasts. But higher spin field theory may become a fashionable topic if a breakthrough happens in understanding its basic problems.

Our experience in quantum field theory teaches us that massive fields with spin 1 and higher are non-renormalizable unless their mass was generated as a result of spontaneous breaking of a gauge symmetry associated with corresponding massless gauge fields. So first of all we should understand the structure of the theory of massless higher spin fields.

In String Theory higher spin excitations become massless in the limit of zero string tension. Thus in this limit one should observe an enhancement of String Theory symmetry by that of the massless higher spin fields, and one can regard string tension generation as a mechanism of spontaneous breaking of the higher spin symmetry. If the conjecture that String Theory is a spontaneously broken phase of an underlying gauge theory of higher spin fields is realized, it can be useful for better understanding of string/M theory and of the (A)dS/CFT correspondence (see e.g. [1, 3, 4, 5, 6, 7] and references therein). This is one of the motivations of the development of the theory of interacting higher spin fields.

A direct but, perhaps, too involved way of studying the higher spin string states would be the one in the framework of String Field Theory, which itself is still under construction as far as supersymmetric and closed strings are concerned. Another possible way is, as in the case of lower spin excitations, to derive an effective field theory of higher spins and to study its properties using conventional field theoretical methods.

In fact, higher spin field theory, both for massive and massless fields, has been developed quite independently of String Theory for a long period of time starting from papers by Dirac [8], Wigner [9], Fierz and Pauli [10], Rarita and Schwinger [11], Bargmann and Wigner [12], Fronsdal [13], Weinberg [14] and others. In last decades a particular attention has been paid to massless higher spin fields whose study revealed a profound and rich geometrical and group-theoretical (conformal) structure underlying their dynamics.

Understanding the interactions of higher spin fields is a main long standing problem of the construction of the higher spin field theory. The interaction problem already reveals

itself when one tries to couple higher spin fields to an electromagnetic field [10, 16] or to gravity [17, 18], or to construct (three-vertex) self-interactions [19, 20, 21]. In the case of massless higher spin fields the problem is in introducing interactions in such a way that they do not break (but may only properly modify) gauge symmetries of the free higher spin field theory. Otherwise the number of degrees of freedom in the interacting theory would differ from that of the free theory, which apparently would result in inconsistencies.

One should also note that the general (Coleman–Mandula and Haag–Lopuszanski–Sohnius) theorem of the possible symmetries of the unitary S-matrix of the quantum field theory in  $D = 4$  Minkowski space [23] does not allow conserved currents associated with symmetries of fields with spin greater than two to contribute to the S-matrix. This no-go theorem might be overcome if the higher spin symmetries would be spontaneously broken, as probably happens in String Theory.

Another way out is that one should construct the interacting higher spin field theory in a vacuum background with a non-zero cosmological constant, such as the Anti de Sitter space, in which case the S-matrix theorem does not apply. This has been realized in [24], where consistent interactions of massless higher spin fields with gravity were constructed in the first non-trivial (cubic) order. Until now the extension of these results to higher orders in the coupling constant at the level of the action has encountered difficulties of a group-theoretical and technical nature related to the problem of finding the full algebraic structure of interacting higher spin symmetries. As has been noted in [20, 21, 24, 25], such an algebraic structure and consistent interactions should involve higher derivative terms and infinite tower of fields with increasing spins, and this again resembles the situation which we have in String Theory. At the level of so called unfolded equations of motion non-linear gauge field models of interacting massless higher spin fields have been constructed in [26, 2, 27].

To study the relation of higher spin field theory to superstring theory one should work in ten-dimensional space-time. Here we encounter a “technical” problem. In  $D = 4$  all states of higher spin can be described either by the higher rank symmetric tensors or spin tensors, since all tensor fields with mixed, symmetric and antisymmetric, components can be related via Poincaré duality to the symmetric tensors. This is not the case, for instance, in  $D = 10$  where mixed symmetry tensor fields describe independent higher spin modes and should be studied separately [28]. From the group-theoretical point of view this is related to the fact that in  $D = 4$  the compact subgroup of the Wigner little group, which is used to classify all the massless irreducible representations of the Poincaré group, is  $SO(2)$  whose Young tableaux are single symmetric rows<sup>1</sup>, while in  $D = 10$  the compact subgroup of the little group is  $SO(8)$  whose representations are described by Young tableaux with both (symmetric) rows and (antisymmetric) columns. An essential progress in studying the mixed symmetry fields has been made only quite recently [29]–[36].

In these lectures, with the purpose of simplifying a bit the comprehension of the material, we shall mainly deal with higher spin fields described by symmetric tensors and spin

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<sup>1</sup>In the case of the massive higher spin fields in  $D = 4$  the Wigner little group is  $SU(2) \sim SO(3)$ , whose irreducible representations are also described by only single row Young tableaux because of the degeneracy of the antisymmetric three-dimensional matrix.

tensors. The article is organized according to its Contents.

These notes are not a comprehensive review but rather an attempt to write an elementary introduction to only few aspects of higher spin field theory and its history. I apologize to the authors whose work has not been reflected in what follows.

## 2. Free higher spin field theory

### 2.1 The choice of Lorentz representations for describing higher spin fields. Symmetric tensors and spin-tensors

One of the possible choices is to associate potentials of integer higher spin fields with symmetric tensors. In  $D = 4$  the symmetric tensors describe all possible higher spin representations of the Poincaré group because the antisymmetric second rank potentials are dual to scalar fields and the three and four form potentials do not carry physical degrees of freedom. As we have already mentioned, this can also be understood using the fact that the compact subgroup of the little group of the  $D = 4$  Lorentz group is one-dimensional  $SO(2)$ . In higher dimensions, for instance in  $D = 10$ , the symmetric tensors do not embrace all the integer higher spins, and one should also consider tensors with the indices of mixed symmetry (symmetric and antisymmetric).

We shall restrict ourselves to the consideration of the symmetric tensor fields  $\phi_{m_1 \dots m_s}(x)$  which, under some conditions to be discussed below, describe higher spin states of an integer spin  $s$ . To describe the physical states of half integer spin  $s$  one should consider spinor tensor fields  $\psi_{m_1 \dots m_{s-\frac{1}{2}}}^\alpha(x)$ <sup>2</sup>.

In string theory symmetric tensor fields arise, for examples, as string states obtained by acting on the vacuum by a single string oscillator  $a_m^{-n}$  with fixed integer  $n$  (e.g.  $n=1$ )

$$\phi_{m_1 \dots m_s} = a_{m_1}^{-1} \dots a_{m_s}^{-1} |0\rangle.$$

Alternatively, the field strengths of half integer and integer spin can be described by symmetric spin-tensors  $\varphi_{\alpha_1 \dots \alpha_{2s}}(x)$  depending on whether  $s$  is half integer or integer [14]. The advantage of this formulation is that all the spins (integer and half integer) are treated on an equal footing.

In the Green-Schwarz formulation of the superstring such fields arise as string states obtained by acting on the string vacuum with an antisymmetrized product of different fermionic oscillators  $\theta_\alpha^{-n}$

$$\varphi_{\alpha_1 \dots \alpha_{2s}} = \theta_{\alpha_1}^{[-1} \dots \theta_{\alpha_{2s}}^{-2s]} |0\rangle,$$

where  $n = 1, \dots, 2s$  labels fermionic oscillator modes.

We shall briefly consider the spin-tensor formulation in Subsection 3.4.

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<sup>2</sup>This formulation of the higher spin fields is also called the metric-like formulation because it is constructed as a generalization of the metric  $\phi_{mn} = g_{mn}$  formulation of General Relativity. For non-linear extensions of higher spin field theories another formalism has proved to be useful. It is based on a generalization of the description of gravity in terms the vielbein and spin connection, and is called the frame-like formulation (see [40] for a review and references).

## 2.2 Symmetric tensor description of massless higher spin fields

As it has already been mentioned, in quantum field theory massive fields with spin 1 and higher are not renormalizable unless their mass is generated as a result of spontaneous breaking of a gauge symmetry associated with corresponding massless gauge fields. So first of all we should understand the structure of the theory of massless higher spin fields and I shall concentrate on this problem. The theory of massive higher spin fields and their interactions (in particular with electromagnetic fields and gravity) was discussed e.g. in [9, 10, 11, 13, 14, 15, 16, 22, 34, 37, 38] .

Note that in  $D = 4$  the physical fields of spin  $s \leq 2$  are part of the family of the symmetric (spin) tensors. Their well known equations of motion and gauge transformations are reproduced below in a form suitable for the generalization to the case of the higher spin fields

$$\begin{aligned}
\mathbf{s=0} \quad & \phi(x) - \text{scalar field, } \partial_m \partial^m \phi \equiv \partial^2 \phi = 0, \text{ matter field, no gauge symmetry;} \\
\mathbf{s = \frac{1}{2}} \quad & \psi^\alpha(x) - \text{spinor field, } \gamma^{m\alpha}_\beta \partial_m \psi^\beta = (\not{\partial} \psi)^\alpha = 0, \text{ matter field, no gauge symmetry;} \\
\mathbf{s=1} \quad & \phi_m(x) = A_m(x) - \text{Maxwell field, } \partial^m F_{mn} = \partial^2 A_n - \partial_n \partial_m A^m = 0, \delta A_m = \partial_m \xi(x); \\
\mathbf{s = \frac{3}{2}} \quad & \psi_m^\alpha(x) - \text{Rarita-Schwinger field, } \gamma_{mnp} \partial^n \psi^p = \not{\partial} \psi_m - \partial_m \gamma^n \psi_n = 0, \delta \psi_m^\alpha = \partial_m \xi^\alpha(x); \\
\mathbf{s=2} \quad & \phi_{m_1 m_2}(x) = g_{m_1 m_2}(x) - \text{graviton, } R_{m_1 m_2} = 0, \delta g_{m_1 m_2} = D_{m_1} \xi_{m_2} + D_{m_2} \xi_{m_1}, \\
& \text{where } D_m = \partial_m + \Gamma_{mn}^p \text{ is the covariant derivative and } \Gamma_{mn,p} = \frac{1}{2}(\partial_p g_{mn} - \partial_m g_{np} - \partial_n g_{mp}) \text{ is the Christoffel connection;} \\
& \text{in the linearized limit where the deviation of } g_{m_1 m_2}(x) \text{ from the Minkowski metric } \eta_{m_1 m_2} \text{ is infinitesimal the Einstein equation and the diffeomorphisms reduce to} \\
& \partial^2 g_{m_1 m_2} - \partial_{m_1} \partial_n g_{m_2}^n - \partial_{m_2} \partial_n g_{m_1}^n + \partial_{m_1} \partial_{m_2} g_n^n = 0, \delta g_{m_1 m_2} = \partial_{m_1} \xi_{m_2} + \partial_{m_2} \xi_{m_1}.
\end{aligned}$$

Except for the scalar and the spinor field, all other massless fields are gauge fields. The associated gauge symmetry eliminates (unphysical) lower spin components of these fields and thus ensures that they have a positive norm. So it is natural to assume that all massless higher spin fields are also the gauge fields with the gauge transformations being an appropriate generalization of those of the Maxwell, Rarita-Schwinger and Einstein field. In the linear (free field) approximation the higher spin gauge transformations (for the integer and half integer spins) have the form

$$\begin{aligned}
\delta \phi_{m_1 \dots m_s}(x) &= \partial_{m_1} \xi_{m_2 \dots m_s} + \partial_{m_2} \xi_{m_1 \dots m_s} + \dots \equiv \sum \partial_{m_1} \xi_{m_2 \dots m_s}, \\
\delta \psi_{m_1 \dots m_{s-\frac{1}{2}}}^\alpha(x) &= \sum \partial_{m_1} \xi_{m_2 \dots m_{s-\frac{1}{2}}}^\alpha,
\end{aligned} \tag{2.1}$$

where  $\sum$  will denote (almost everywhere) the symmetrized sum with respect to all non-contracted vector indices.

### 2.2.1 Free equations of motion

We assume that the free equations of motion of the higher spin fields are second order linear differential equations in the case of the integer spins and the first order differential

equations in the case of the half integer spins. This is required by the unitarity and ensures that the fields have a positive-definite norm. The massless higher spin equations have been derived from the massive higher spin equations [16] by Fronsdal for bosons [41] and by Fang and Fronsdal for fermions [42], and studied in more detail in [44].

The bosonic equations, which I shall denote by  $G_{m_1 \dots m_s}(x)$  are a natural generalization of the Klein-Gordon, Maxwell and linearized Einstein equations

$$G_{m_1 \dots m_s}(x) \equiv \partial^2 \phi_{m_1 \dots m_s}(x) - \sum \partial_{m_1} \partial_n \phi_{m_2 \dots m_s}^n(x) + \sum \partial_{m_1} \partial_{m_2} \phi_{nm_3 \dots m_s}^n(x) = 0. \quad (2.2)$$

The first order fermionic equations are a natural generalization of the Dirac and Rarita-Schwinger equation

$$G_{m_1 \dots m_{s-\frac{1}{2}}}^\alpha(x) \equiv (\not{\partial} \psi)_{m_1 \dots m_{s-\frac{1}{2}}}^\alpha - \sum \partial_{m_1} (\gamma^n \psi)_{nm_2 \dots m_{s-\frac{1}{2}}}^\alpha = 0. \quad (2.3)$$

### 2.2.2 Constraints on higher spin symmetry parameters and on higher spin fields

We should now verify that the equations of motion (2.2) and (2.3) are invariant under gauge transformations (2.1). The direct computations give

$$\delta G_{m_1 \dots m_s} = 3 \sum \partial_{m_1 m_2 m_3}^3 \xi_{nm_4 \dots m_s}^n, \quad \delta G_{m_1 \dots m_{s-\frac{1}{2}}}^\alpha = -2 \sum \partial_{m_1 m_2}^2 \gamma^{n\alpha}{}_\beta \xi_{nm_3 \dots m_{s-\frac{1}{2}}}^\beta, \quad (2.4)$$

where  $\partial_{m_1 m_2}^2 = \partial_{m_1} \partial_{m_2}$  and  $\partial_{m_1 m_2 m_3}^3 = \partial_{m_1} \partial_{m_2} \partial_{m_3}$ .

We see that these variations vanish if the parameters of the transformations of the bosonic higher spin fields (for  $s \geq 3$ ) are traceless

$$\xi_{nm_4 \dots m_s}^n = 0 \quad (2.5)$$

and the parameters of the transformations of the fermionic higher spin fields for ( $s \geq 5/2$ ) are  $\gamma$ -traceless

$$(\gamma^n \xi)_{nm_3 \dots m_{s-\frac{1}{2}}}^\alpha = 0. \quad (2.6)$$

Other constraints in the theory of higher spins appear for bosons with  $s \geq 4$  and for fermions with  $s \geq \frac{7}{2}$ . Since the theory is gauge invariant there should exist Bianchi identities analogous to those in Maxwell and Einstein theory which are identically satisfied. The Bianchi identities (or, equivalently, integrability conditions) imply that the traceless divergence of the left-hand-side of equations of motion must vanish identically. This also implies that the currents of the matter fields if coupled to the gauge fields are conserved.

For instance in Maxwell theory we have

$$\partial_n (\partial_m F^{mn}) \equiv 0, \quad (2.7)$$

and, hence the electric current which enters the r.h.s. of the Maxwell equations  $\partial_m F^{mn} = J^m$  is conserved  $\partial_m J^m = 0$ .

In the theory of gravity coupled to matter fields and described by the Einstein equation

$$R_{mn} - \frac{1}{2} g_{mn} R = T_{mn}$$

the energy–momentum conservation  $D_m T^{mn} = 0$  is related to the Bianchi identity

$$D_m R_n^m - \frac{1}{2} D_n R_m^m \equiv 0. \quad (2.8)$$

The linearized form of (2.8) generalized to the case of the bosonic higher spin fields results in the following Bianchi identity (or the integrability condition)

$$\partial_n G_{m_2 \dots m_s}^n - \frac{1}{2} \sum \partial_{m_2} G_{nm_3 \dots m_s}^n = -\frac{3}{2} \sum \partial_{m_2 m_3 m_4}^3 \phi_{npm_5 \dots m_s}^{np}, \quad (2.9)$$

and in the case of the fermionic fields we have

$$\partial_n G_{m_2 \dots m_{s-\frac{1}{2}}}^{\alpha n} - \frac{1}{2} \sum \partial_{m_2} G_{nm_3 \dots m_{s-\frac{1}{2}}}^{\alpha n} - \frac{1}{2} (\not{\partial} \gamma^n G)_{nm_2 \dots m_{s-\frac{1}{2}}}^\alpha = \sum \partial_{m_2 m_3}^2 (\gamma^n \psi)^{\alpha p}_{npm_4 \dots m_{s-\frac{1}{2}}}. \quad (2.10)$$

We see that the right–hand–sides of (2.9) and (2.10) do not vanish identically and require that the bosonic fields with  $s \geq 4$  are double–traceless

$$\phi_{npm_5 \dots m_s}^{np} = 0 \quad (2.11)$$

and the fermionic fields with  $s \geq \frac{7}{2}$  are triple–gamma–traceless

$$(\gamma^n \gamma^p \gamma^r \psi)^\alpha_{npr m_4 \dots m_{s-\frac{1}{2}}} \equiv (\gamma^n \psi)^{\alpha p}_{npm_4 \dots m_{s-\frac{1}{2}}} = 0. \quad (2.12)$$

It turns out that for the consistency of the theory the fields should satisfy the double–triple traceless conditions identically, i.e. *off the mass shell*. Note that the double– and triple–traceless conditions are the strongest possible gauge invariant algebraic constraints on the fields, provided that the gauge parameters are traceless.

Physically the requirement of the double tracelessness, together with the gauge fixing of higher spin symmetry, ensures that the lower spin components contained in the symmetric tensor fields are eliminated, so that only the massless states with helicities  $\pm s$  propagate. And as we know very well for lower spin fields  $\frac{1}{2} \leq s \leq 2$ , in  $D = 4$  each massless field has only two physical degrees of freedom which are characterized by the helicities  $\pm s$ .

Pure gauge degrees of freedom of the integer higher spin fields can be eliminated by imposing gauge fixing conditions analogous to the Lorentz gauge of the vector field and the de Donder gauge in the case of gravity

$$\partial_p \phi_{m_2 \dots m_s}^p - \frac{1}{2} \sum \partial_{m_2} \phi_{pm_3 \dots m_s}^p = 0. \quad (2.13)$$

Then the higher spin equations of motion (2.2) reduce to the Klein–Gordon equation  $\partial^2 \phi_{m_1 \dots m_s} = 0$ , which implies that we indeed deal with *massless* fields.

Covariant gauge fixing condition for fermion fields are [44]

$$\gamma^n \psi_{nm_2 \dots m_{s-\frac{1}{2}}} = 0, \quad \Rightarrow \quad \psi_{nm_3 \dots m_{s-\frac{1}{2}}}^n = 0. \quad (2.14)$$

They reduce the field equation (2.3) down to the massless Dirac equation.

Thus, the double–and triple–traceless constraints along with the gauge fixing conditions single out physical components of the massless higher spin fields. Another role of the



double– and triple–traceless constraints (2.11), (2.12) is that only when the fields identically satisfy (2.11) and (2.12), the higher spin field equations (2.2) and (2.3) can be obtained from appropriate actions [41, 44].

Unconstrained formulations of higher spin field dynamics will be considered in the next Section.

### 2.2.3 The free higher spin field actions

For the bosonic fields the action in  $D$ –dimensional space–time is

$$S_B = \int d^D x \left( \frac{1}{2} \phi^{m_1 \dots m_s} G_{m_1 \dots m_s} - \frac{1}{8} s(s-1) \phi_n^{nm_3 \dots m_s} G_{pm_3 \dots m_s}^p \right) \quad (2.15)$$

and for fermions

$$S_F = \int d^D x \left( -\frac{1}{2} \bar{\psi}^{m_1 \dots m_{s-\frac{1}{2}}} G_{m_1 \dots m_{s-\frac{1}{2}}} + \frac{1}{4} s \bar{\psi}^{m_2 \dots m_{s-\frac{1}{2}}} \gamma_n \gamma^p G_{pm_2 \dots m_{s-\frac{1}{2}}} \right. \\ \left. + \frac{1}{8} s(s-1) \bar{\psi}_n^{nm_3 \dots m_{s-\frac{1}{2}}} G_{pm_3 \dots m_{s-\frac{1}{2}}}^p \right), \quad (2.16)$$

where  $G_{m_1 \dots m_s}$  and  $G_{m_1 \dots m_{s-\frac{1}{2}}}^\alpha$  stand for the left hand sides of the equations (2.2) and (2.3).

The actions are invariant under the gauge transformations (2.1) with the traceless parameters (2.5) and (2.6), and the higher spin fields are supposed to be double or gamma–triple traceless (2.11), (2.12).

## 3. Geometric aspects of free higher spin field theory

The presence of the constraints on the gauge parameters and higher spin fields in the formulation of Fronsdal and of Fang and Fronsdal may look as an odd feature of the theory and point out that such a formulation is incomplete. A modification of the equations of motion (2.2) and (2.3) which would remove the constraints (2.5) and (2.6) on the gauge parameters and the double–traceless conditions (2.11), (2.12) can be achieved in three different though related ways.

One of the ways to remove the tracelessness constraints is to use, in addition to the physical higher spin field, an appropriate number of auxiliary tensorial fields satisfying certain equations of motion, as was shown in [29]<sup>3</sup>. The higher spin field equations remain lagrangian [29, 39].

Another way was proposed by Francia and Sagnotti [45]. Its key point is to renounce locality of the theory. It was shown that the equations of motion of the unconstrained higher spin fields and corresponding actions can be made invariant under the unconstrained gauge transformations if they are enlarged with *non–local* terms. A motivation of Francia and Sagnotti for removing constraints on gauge parameters has been based on the observation that symmetries of String Field Theory do not have such restrictions. Another motivation

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<sup>3</sup>In the case of massive higher spin fields, auxiliary fields to construct higher spin field actions were introduced by Fierz and Pauli [10].

was to find a more conventional *geometric* form of the higher-spin field equations in terms of conditions on generalized curvatures introduced in [44]. This would be a generalization of the Maxwell and Einstein equations written in terms of  $F_{mn}$  and  $R_{mn}$ , respectively. The choice of non-local terms in the higher spin field equations is not unique. Choosing a suitable form of non-local equations one manages to keep their Lagrangian nature. We will not go into details of this formulation and address the interested reader to [45, 46].

A third possibility of removing the constraints is to allow the higher spin field *potentials* to satisfy *higher order* differential equations, which can be constructed in a manifestly gauge invariant way as conditions imposed on the higher spin field curvatures. Let us consider this geometric formulation of the free higher spin field theory in more detail. Actually, in  $D = 4$  space-time it has been constructed many years ago by Bargmann and Wigner [12]. As we shall see, the higher order derivative structure of the higher spin curvature equations does not spoil the unitarity of the theory. These equations are physically equivalent to the Fang-Fronsdal and Francia-Sagnotti equations.

Generalized curvatures for the higher spin fields  $\phi_{m_1 \dots m_s}(x)$  and  $\psi_{m_1 \dots m_{s-\frac{1}{2}}}^\alpha(x)$  which are invariant under the *unconstrained* gauge transformations (2.1) can be constructed as a direct generalization of the spin 1 Maxwell field strength

$$F_{mn} = \partial_m A_n - \partial_n A_m$$

and of the linearized Riemann tensor in the case of spin 2

$$R_{m_1 n_1, m_2 n_2} = \partial_{m_1} \partial_{m_2} g_{n_1 n_2} - \partial_{n_1} \partial_{m_2} g_{m_1 n_2} - \partial_{m_1} \partial_{n_2} g_{n_1 m_2} + \partial_{n_1} \partial_{n_2} g_{m_1 m_2}. \quad (3.1)$$

Thus, for an arbitrary integer spin  $s$  the gauge invariant curvature is obtained by taking  $s$  derivatives of the field potential  $\phi_{n_1 \dots n_s}(x)$

$$\begin{aligned} R_{m_1 n_1, m_2 n_2, \dots, m_s n_s} &= \partial_{m_1} \partial_{m_2} \dots \partial_{m_s} \phi_{n_1 n_2 \dots n_s} - \partial_{n_1} \partial_{m_2} \dots \partial_{m_s} \phi_{m_1 n_2 \dots n_s} \\ &\quad - \partial_{m_1} \partial_{n_2} \dots \partial_{m_s} \phi_{n_1 m_2 \dots n_s} + \partial_{n_1} \partial_{n_2} \dots \partial_{m_s} \phi_{m_1 m_2 \dots n_s} + \dots \\ &\equiv \partial_{m_1 \dots m_s}^s \phi_{n_1 \dots n_s} - \sum (m_i \leftrightarrow n_i). \end{aligned} \quad (3.2)$$

Analogously, for an arbitrary half integer spin  $s$  the curvature is obtained by taking  $(s - \frac{1}{2})$  derivatives of the field potential  $\psi_{n_1 \dots n_{s-\frac{1}{2}}}^\alpha(x)$

$$\begin{aligned} \mathcal{R}_{m_1 n_1, m_2 n_2, \dots, m_{s-\frac{1}{2}} n_{s-\frac{1}{2}}}^\alpha &= \partial_{m_1} \partial_{m_2} \dots \partial_{m_{s-\frac{1}{2}}} \psi_{n_1 n_2 \dots n_{s-\frac{1}{2}}}^\alpha - \partial_{n_1} \partial_{m_2} \dots \partial_{m_{s-\frac{1}{2}}} \psi_{m_1 n_2 \dots n_{s-\frac{1}{2}}}^\alpha \\ &\quad - \partial_{m_1} \partial_{n_2} \dots \partial_{m_{s-\frac{1}{2}}} \psi_{n_1 m_2 \dots n_{s-\frac{1}{2}}}^\alpha + \partial_{n_1} \partial_{n_2} \dots \partial_{m_{s-\frac{1}{2}}} \psi_{m_1 m_2 \dots n_{s-\frac{1}{2}}}^\alpha + \dots \\ &\equiv \partial_{m_1 \dots m_{s-\frac{1}{2}}}^{s-\frac{1}{2}} \psi_{n_1 \dots n_{s-\frac{1}{2}}}^\alpha - \sum (m_i \leftrightarrow n_i). \end{aligned} \quad (3.3)$$

In the right hand side of (3.2) and (3.3) it is implied that the sum is taken over all the terms in which the indices within the pairs of  $[m_i, n_i]$  with the same label  $i = 1, \dots, s$  are antisymmetrized.

By construction similar to the Riemann tensor (3.1), the higher spin curvatures (3.2) and (3.3) are completely symmetric under the exchange of any two pairs of their antisymmetric indices and they obey for any pair of the antisymmetric indices  $[m_i n_i]$  the same Bianchi identities as the Riemann tensor, *e.g.* for the bosonic spin  $s$  field

$$R_{m_1 n_1, m_2 n_2, \dots, m_s n_s} = -R_{n_1 m_1, m_2 n_2, \dots, m_s n_s} = R_{m_2 n_2, m_1 n_1, \dots, m_s n_s}, \quad (3.4)$$

$$R_{[m_1 n_1, m_2] n_2, \dots, m_s n_s} = 0, \quad (3.5)$$

$$\partial_{[l_1} R_{m_1 n_1], m_2 n_2, \dots, m_s n_s} = 0. \quad (3.6)$$

On the other hand, if a rank  $2[s]$  (spinor)–tensor (where  $[s]$  is the integer part of  $s$ ) possesses the properties (3.4)–(3.6), in virtue of the generalized Poincaré lemma of [47, 48, 49] this tensor can be expressed as an ‘antisymmetrized’  $[s]$ –th derivative of a symmetric rank  $[s]$  field potential, as in eqs. (3.2) and (3.3).

Let us note that de Wit and Freedman [44] constructed curvature tensors out of the  $[s]$  derivatives of the symmetric higher spin gauge fields  $\phi_{m_1 \dots m_s}(x)$  and  $\psi_{m_1 \dots m_{s-\frac{1}{2}}}^\alpha(x)$  in an alternative way. Their curvatures have two pairs of the groups of  $[s]$  symmetric indices and they are symmetric or antisymmetric under the exchange of these groups of indices depending on whether  $[s]$  is even or odd

$$\tilde{R}_{m_1 \dots m_{[s]}, n_1 \dots n_{[s]}} = (-1)^{[s]} \tilde{R}_{n_1 \dots n_{[s]}, m_1 \dots m_{[s]}}. \quad (3.7)$$

The de Wit–Freedman curvatures satisfy the cyclic identity, which is a symmetric analog of (3.5),

$$\tilde{R}_{m_1 \dots m_{[s]}, n_1 \dots n_{[s]}} + \sum_{n_i} \tilde{R}_{n_1 m_2 \dots m_{[s]}, m_1 n_2 \dots n_{[s]}} = 0, \quad (3.8)$$

where  $\sum_{n_i}$  denotes the symmetrized sum with respect to the indices  $n_i$ . The form of the analogs of the differential Bianchi identities (3.6) for  $\tilde{R}_{m_1 \dots m_{[s]}, n_1 \dots n_{[s]}}$  is less transparent. They can be obtained from (3.6) using the fact that the de Wit–Freedman tensors (3.7) are related to the generalized ‘Riemann’ tensors (3.2) and (3.3) by the antisymmetrization in the former of the indices of each pair  $[m_i, n_i]$ . In what follows we shall work with the generalized Riemann curvatures.

To produce the dynamical equations of motion of the higher spin fields one should now impose additional conditions on the higher spin curvatures. Since the curvatures (3.2) and (3.3) have the same properties (3.4)–(3.6) as the Riemann tensor (3.1), and the linearized Einstein equation amounts to putting to zero the trace of the Riemann tensor

$$R^m_{n_1, m n_2} = R_{n_1 n_2} = \partial^2 \partial^2 g_{n_1 n_2} - \partial_{n_1} \partial_m g^m_{n_2} - \partial_{n_2} \partial_m g^m_{n_1} + \partial_{n_1} \partial_{n_2} g^m_m = 0, \quad (3.9)$$

one can naturally assume that in the case of the integer spins the equations of motion are also obtained by requiring that the trace of the generalized Riemann tensor (3.2) with respect to any pair of indices is zero, *e.g.*

$$R^m_{n_1, m n_2, m_3 n_3, \dots, m_s n_s} = 0. \quad (3.10)$$

In the case of the half integer spins we can assume that the fermionic equations of motion are a generalization of the Dirac and Rarita–Schwinger equations and that they are obtained by putting to zero the gamma–trace of the fermionic curvature (3.3)

$$(\gamma^{m_1} \mathcal{R})_{m_1 n_1, m_2 n_2, \dots, m_{s-\frac{1}{2}} n_{s-\frac{1}{2}}}^\alpha = 0. \quad (3.11)$$

The equations (3.10) and (3.11) are non–lagrangian for  $s > 2$ .

Let us remind the reader that, because of the gauge invariance of the higher spin curvatures, eqs. (3.10) and (3.11) are invariant under the higher spin gauge transformations (2.1) with *unconstrained* parameters, and the higher spin field potentials are also *unconstrained* in contrast to the Fronsdal and Fang–Fronsdal formulation considered in the previous Section 2.1.

The reason and the price for this is that eqs. (3.10) and (3.11) are higher order differential equations, which might cause a problem with unitarity of the quantum theory. However, as we shall now show the equations (3.10) and (3.11) reduce to, respectively, the second and first order differential equations for the higher spin field potentials related to those of Fronsdal (2.2) and of Fang and Fronsdal (2.3).

### 3.1 Integer spin fields

In the integer spin case, analyzing the form of the left hand side of eq. (3.10) in terms of the gauge field potential (3.2) one gets the higher spin generalization of the spin 3 Damour–Deser identity [50] which relates the trace of the higher spin curvature to the left hand side of the Fronsdal equations, namely

$$\begin{aligned} \text{tr } R_{m_1 n_1, \dots, m_s n_s} &= R_{m_1 n_1, m_2 n_2, \dots, n_{s-1} m, n_s}^m \\ &= \partial_{m_1 m_2 \dots m_{s-2}}^{s-2} G_{n_1 \dots n_{s-2} n_{s-1} n_s} - \sum_{i=1}^{i=s-2} (m_i \leftrightarrow n_i) \end{aligned} \quad (3.12)$$

where the symmetric tensor  $G(x)$  stands for the left hand side of the Fronsdal equations (2.2) (sometimes called the “Fronsdal kinetic operator”), and the indices  $[m_i, n_i]$  with  $(i = 1, \dots, s-2)$  are anti–symmetrized.

When the curvature tensor satisfies the tracelessness condition (3.10) the left hand side of eq. (3.12) vanishes, which implies that the tensor  $G$  is  $\partial^{s-2}$ –closed. In virtue of the generalized Poincaré lemma [47, 48, 49] this means that (at least locally)  $G$  is  $\partial^3$ –exact, *i.e.* has the form [51]

$$G_{n_1 \dots n_s} = \sum \partial_{n_1 n_2 n_3}^3 \rho_{n_4 \dots n_s}, \quad (3.13)$$

where the sum implies the symmetrization of all the indices  $n_i$  and  $\rho(x)$  is a symmetric tensor field of rank  $(s-3)$  called ‘compensator’ since its gauge transformation

$$\delta \rho_{n_1 \dots n_{s-3}} = 3 \xi_{mn_1 \dots n_{s-3}}^m \quad (3.14)$$

compensates the non–invariance (2.4) of the kinetic operator  $G(x)$  under the unconstrained local variations (2.1) of the gauge field potential  $\phi(x)$ . Eq. (3.13) was discussed in [45, 70]. Here we have obtained it from the geometric equation on the higher spin field curvature

(3.10) following ref. [51] where such a derivation was carried out for a generic mixed symmetry field <sup>4</sup>.

The trace of the gauge parameter (3.14) can be used to eliminate the compensator field. Then eq. (3.13) reduces to the Fronsdal equation (2.2) which is invariant under residual gauge transformations with traceless parameters.

### 3.2 A non-local form of the higher spin equations

We shall now demonstrate how the higher spin field equations with the compensator (3.13) are related to the non-local equations of [45]. We shall consider the simple (standard) example of a gauge field of spin 3. The case of a generic spin  $s$  can be treated in a similar but more tedious way. In a somewhat different way the relation of the compensator equations (3.13) to non-local higher spin equations was discussed in the second paper of [45].

For the *spin* 3 field the compensator equation takes the form

$$G_{mnp} := \partial^2 \phi_{mnp} - 3\partial_q \partial_{(m} \phi_{np)}^q + 3\partial_{(m} \partial_n \phi_{p)q}^q = \partial_{mnp}^3 \rho(x), \quad (3.15)$$

where  $()$  stand for the symmetrization of the indices with weight one and  $\rho(x)$  is the compensator, which is a scalar field in the case of spin 3.

We now take the derivative and then the double trace of the left and the right hand side of this equation and get

$$\partial_m G^{mn}{}_n = \partial^2 \partial^2 \rho(x). \quad (3.16)$$

Modulo the *doubly harmonic* zero modes  $\rho_0(x)$ , satisfying  $\partial^2 \partial^2 \rho_0(x) = 0$ , one can solve eq. (3.16) for  $\rho(x)$  in a non-local form

$$\rho(x) = \frac{1}{\partial^2 \partial^2} \partial_m G^{mn}{}_n. \quad (3.17)$$

Substituting this solution into the spin 3 field equation (3.15) we get one of the non-local forms of the spin 3 field equation constructed in [45]

$$G_{mnp} := \partial^2 \phi_{mnp} - 3\partial_q \partial_{(m} \phi_{np)}^q + 3\partial_{(m} \partial_n \phi_{p)q}^q = \frac{1}{\partial^2 \partial^2} \partial_{mnp}^3 (\partial_q G^{qr}{}_r). \quad (3.18)$$

Let us now consider a more complicated example of *spin* 4. In the Fronsdal formulation, the fields of spin 4 and higher feature one more restriction: they are double traceless. We shall show how this constraint appears upon gauge fixing the compensator equation, which for the spin 4 field has the form

$$G_{mnpq} := \partial^2 \varphi_{mnpq} - 4\partial_r \partial_{(m} \varphi_{npq)}^r + 6\partial_{(m} \partial_n \varphi_{pq)r}^r = 4\partial_{(mnp}^3 \rho_q(x). \quad (3.19)$$

Taking the double trace of (3.19) we have

$$G^{mn}{}_{mn} = 3\partial^2 \varphi^{mn}{}_{mn} = 4\partial^2 \partial_m \rho^m. \quad (3.20)$$

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<sup>4</sup>For a relevant earlier discussion of the relationship of the equations on the Riemann and de Wit-Freedman curvatures to the equations of motion of symmetric (spinor)-tensor field potentials see [52].

Taking the derivative of (3.19) and the double trace we get

$$\partial_m G^{mn}_{np} = \partial^2 \partial^2 \rho_p + 3 \partial_p \partial^2 \partial_m \rho^m = \partial^2 \partial^2 \rho_p + \frac{3}{4} \partial_p G^{mn}_{mn}, \quad (3.21)$$

where we have used (3.20) to arrive at the right hand side of (3.21).

From (3.21) we find that, modulo the zero modes  $\rho_0^p$  of  $\partial^2 \partial^2 \rho_0^p = 0$ , the compensator field is non-locally expressed in terms of the (double) trace of the Fronsdal kinetic term

$$\rho_p = \frac{1}{\partial^2 \partial^2} (\partial_m G^{mn}_{np} - \frac{3}{4} \partial_p G^{mn}_{mn}). \quad (3.22)$$

Substituting (3.22) into (3.19) we get one of the non-local forms [45] of the spin 4 field equation [45].

Consider now the following identity

$$\partial_q G^q_{mnp} - \partial_{(m} G^q_{np)q} = -\frac{3}{2} \partial_m \partial_n \partial_p \phi^{qr}_{qr} = -2 \partial_m \partial_n \partial_p (\partial_q \rho^q). \quad (3.23)$$

On the other hand, from (3.20) and (3.23) it follows that modulo a linear and quadratic term in  $x^m$  (which can be put to zero by requiring an appropriate asymptotic behaviour of the wave function at infinity) the double trace of the gauge field  $\phi(x)$  is proportional to the divergence of  $\rho_q(x)$

$$\phi^{mn}_{mn} = \frac{4}{3} \partial_q \rho^q. \quad (3.24)$$

Therefore, when we partially fix the gauge symmetry by putting  $\rho_q(x) = 0$ , the double trace of the gauge field also vanishes and we recover the Fronsdal formulation with the traceless gauge parameter and the double traceless gauge field.

### 3.3 Half integer spin fields

Let us generalize the previous consideration to the case of fermions. The fermionic spin- $s$  field strength  $\mathcal{R}^\alpha$  is the spinor-tensor

$$\mathcal{R}^\alpha_{m_1 n_1, \dots, m_{s-\frac{1}{2}} n_{s-\frac{1}{2}}}(x). \quad (3.25)$$

It satisfies the Bianchi identities analogous to (3.5), (3.6) and thus can be expressed in terms of  $s - \frac{1}{2}$  derivatives of a fermionic field potential (3.3).

The fermionic generalization of the Damour-Deser identity is

$$(\gamma^m \mathcal{R})^\alpha_{mn_1, m_2 n_2, \dots, m_{s-\frac{1}{2}} n_{s-\frac{1}{2}}} = \partial_{m_1 \dots m_{s-\frac{3}{2}}}^{s-\frac{3}{2}} G^\alpha_{n_1 \dots n_{s-\frac{3}{2}} n_{s-\frac{1}{2}}} - \sum_{i=1}^{i=s-\frac{3}{2}} (m_i \leftrightarrow n_i), \quad (3.26)$$

where the same is taken over terms with the indices  $[m_i, n_i]$  anti-symmetrized, and the Fang-Fronsdal fermionic kinetic operator  $G^\alpha$  acting on the gauge field  $\psi^\alpha$  is defined in (2.3). The field strength (3.3) is invariant under the *unconstrained* gauge transformations (2.1).

When the fermionic field strength satisfies the  $\gamma$ -tracelessness condition (3.11)

$$(\gamma^m \mathcal{R})_{mn_1, m_2 n_2, \dots, m_{s-\frac{1}{2}} n_{s-\frac{1}{2}}}^\alpha = 0,$$

eq. (3.26) implies that  $G^\alpha$  is  $\partial^{s-\frac{3}{2}}$ -closed. Since  $\partial^{s+\frac{1}{2}} \equiv 0$ , by virtue of the generalized Poincaré lemma  $G^\alpha$  is  $\partial^2$ -exact

$$G^\alpha_{n_1 \dots n_{s-\frac{1}{2}}} = \sum \partial_{n_1} \partial_{n_2} \rho^\alpha_{n_3 \dots n_{s-\frac{1}{2}}}, \quad (3.27)$$

where  $\sum$  implies the symmetrization of all the indices  $n_i$ .

Equation (3.27) is the compensator equation given in [45, 70]. The demonstration of its relation to the gamma-traceless part of the fermionic higher spin field strength has been given in [53].

The gauge variation of  $G^\alpha(x)$  has been presented in (2.4). It is compensated by a gauge shift of the field  $\rho^\alpha(x)$  given by the  $\gamma$ -trace of the gauge parameter

$$\delta \rho^\alpha_{n_3 \dots n_{s-\frac{1}{2}}} = -2 \sum \gamma^{m\alpha}_\beta \xi^\beta_{mn_3 \dots n_{s-\frac{1}{2}}}. \quad (3.28)$$

Thus, the compensator can be gauged away by choosing a gauge parameter  $\xi^\alpha(x)$  with the appropriate  $\gamma$ -trace. Then, the equations of motion of the gauge field  $\psi^\alpha(x)$  become the first order differential equations (2.3) which are invariant under the gauge transformations (2.1) with  $\gamma$ -traceless parameters.

Alternatively, one can get non-local Francia-Sagnotti equations for fermions by taking a particular non-local solution for the compensator field in terms of the fermionic kinetic operator  $G^\alpha$ . As a simple example consider the  $s = 5/2$  case. Eq. (3.27) takes the form

$$G^\alpha_{mn} := \not{\partial} \psi^\alpha_{mn} - 2 \partial_{(m} (\gamma^q \psi)^\alpha_{n)q} = \partial_m \partial_n \rho^\alpha(x). \quad (3.29)$$

Taking the trace of (3.29) we get

$$\partial^2 \rho^\alpha = G^{\alpha p}{}_p. \quad (3.30)$$

Hence, modulo the zero modes  $\rho^\alpha_0(x)$  of the Klein-Gordon operator  $\partial^2 \rho^\alpha_0 = 0$  the compensator field is non-locally expressed in terms of the trace of  $G^\alpha_{mn}$

$$\rho^\alpha = \frac{1}{\partial^2} G^{\alpha p}{}_p. \quad (3.31)$$

Substituting (3.31) into (3.29) we get the Francia-Sagnotti equation for the fermionic field of spin 5/2 in the following form

$$G^\alpha_{mn} := \not{\partial} \psi^\alpha_{mn} - 2 \partial_{(m} (\gamma^q \psi)^\alpha_{n)q} = \frac{1}{\partial^2} \partial_m \partial_n G^{\alpha p}{}_p. \quad (3.32)$$

In the same way one can relate the compensator equations for an arbitrary half integer spin field to the corresponding non-local field equation. As in the bosonic case, one can find that for  $s \geq \frac{7}{2}$  the triple-gamma trace of the fermionic gauge field potential is expressed in terms of the  $\gamma$ -trace and the divergence of the compensator field and thus vanishes in the 'Fronsdal gauge'  $\rho^\alpha = 0$ .

We have thus reviewed various formulations of free higher spin field dynamics.

## 4. The interaction problem

As we have seen, the free theory of higher spin fields, both massless and massive, exists and can be formulated in a conventional field-theoretical fashion using the action principle. An important problem, which still has not been completely solved, is to introduce interactions of the higher spin fields. Probably, String Field Theory should give the answer to this problem if someone manages to extract the corresponding information from the String Field Theory action. This itself is a highly non-trivial problem which has not been realized yet.

So far the study of the problem of higher spin interactions has been undertaken mainly in the framework of the standard field-theoretical approach, and I would now like to review main obstacles which one encounters in the way of constructing an interacting *massless* higher spin field theory.

One may consider self-interactions of fields of the same spin, or interactions among fields of different spin. In the first case, for example, the construction of consistent self-interactions of massless vector fields results in either the non-abelian Yang-Mills theory [21] or in the non-linear Dirac-Born-Infeld generalization of Maxwell theory. A consistent way of introducing the self-interaction of the spin 2 field leads to the Einstein theory of gravity [19]. Consistency basically means that the introduction of interactions should not break the gauge symmetry, but may only modify it in a suitably way.

An example of interactions among fields of different spin is the universal gravitational interaction of the matter and gauge fields. So, the construction of the theory of interactions of higher spin fields with gravity is an important part of the general interaction problem and it actually exhibits all aspects of the general problem.

### 4.1 Simple supergravity

To see what kind of problems with the higher spin interactions arise, let us first consider the example of coupling to four-dimensional gravity the field of spin 3/2 [10, 11] historically called the Rarita-Schwinger field. This is an instructive example which shows how the consistency of gravitational coupling leads to supergravity [54, 55], the theory invariant under local supersymmetry in which the spin 3/2 field becomes the superpartner of the graviton, called gravitino.

The general coordinate invariance of the complete non-linear gravitational interactions requires that in the free field equations partial derivatives get replaced with covariant derivatives and the vector indices are contracted with the gravitational metric  $g_{mn}(x)$ . So the free Rarita-Schwinger equation

$$\gamma^{mnp} \partial_n \psi_p = 0$$

should be generalized to include the interaction with gravity as follows

$$G^m = \gamma^{mnp} D_n \psi_p = 0, \tag{4.1}$$

where  $\gamma_n = e_n^a(x) \gamma_a$  are the gamma-matrices contracted with the vielbein  $e_n^a(x)$  which is related to the metric in the standard way  $g_{mn} = e_m^a e_n^b \eta_{ab}$ , and  $D_m = \partial_m +$



$\Gamma_{mn}^p + \omega_{m\alpha}^\beta$  is the covariant derivative which contains the Christoffel symbol  $\Gamma_{mn}^p$  and a spin connection  $\omega_{m\alpha}^\beta$  acting on spinor indices.

In the presence of the Rarita–Schwinger field the right–hand–side of the Einstein equations acquires the contribution of an energy–momentum tensor of the spin 3/2 field

$$R_{mn} - \frac{1}{2}g_{mn}R = T_{mn}(\psi), \quad \text{or} \quad R_{mn} - T_{mn} + \frac{1}{2}g_{mn}T_l{}^l = 0. \quad (4.2)$$

The explicit form of  $T_{mn}$  is not known until a consistent interacting theory is constructed.

Consider now the variation of the Rarita–Schwinger field equation under the gauge transformations

$$\delta\psi_m^\alpha = D_m\xi^\alpha, \quad (4.3)$$

which are the general covariant extension of the free field gauge transformations. The variation of eq. (4.1) is

$$\delta G^m = \gamma^{mnp}D_n D_p \xi = \frac{1}{2}\gamma^{mnp}[D_n, D_p]\xi \sim \gamma^{mnp}R_{np,qr}\gamma^{qr}\xi \sim R^m{}_n \gamma^n \xi, \quad (4.4)$$

where the commutator of  $D_n$  produces the Riemann curvature which I schematically write as

$$[D_m, D_n] \sim R_{mn,pq}\gamma^{pq}, \quad (4.5)$$

and the last term in (4.4) is obtained by use of  $\gamma$ –matrix identities.

We thus see that the gauge variation of the Rarita–Schwinger equation is proportional to the Ricci tensor. It is zero if the Ricci tensor is zero, i.e. when the gravitational field satisfies the Einstein equations in the absence of the matter fields. This is satisfactory if we are interested in the dynamics of the spin 3/2 field in the external background of a gravitational field, such as free gravitational waves, for example. But if we would like to consider a closed graviton–spin 3/2 system, then the Ricci tensor is non–zero, since the Einstein equations take the form (4.2), and the variation (4.4) does not vanish. To improve the situation we should require that the graviton also non–trivially varies under the gauge transformations with the spinorial parameter  $\xi^\alpha(x)$  as follows

$$\delta g_{mn} = \frac{i}{2}(\bar{\psi}_n \gamma_m \xi + \bar{\psi}_m \gamma_n \xi), \quad (4.6)$$

and we should take into account this variation of the graviton in the Rarita–Schwinger equation. This will result in the following variation of the Rarita–Schwinger equation

$$\delta G_m \sim (R_{mn} - T_{mn} + \frac{1}{2}g_{mn}T_l{}^l)\gamma^n \xi, \quad (4.7)$$

provided we also add appropriate second–order and fourth–order fermionic terms into the definition of the covariant derivative  $D_m$ , into the variation of  $\psi_m$  (4.3) and into the definition of its energy–momentum tensor. We observe that the variation of the Rarita–Schwinger equation has become proportional to the Einstein equation and hence vanishes.

Thus, by modifying the gauge transformations of the Rarita–Schwinger field, of the graviton and by appropriately modifying the Rarita–Schwinger and the Einstein equations

we have achieved that under the gauge (supersymmetry) transformations with the fermionic parameter  $\xi^\alpha$  the Rarita–Schwinger and the Einstein equations transform into each other and hence consistently describe the coupling of the spin-3/2 field to gravity.

What we have actually obtained is a simple  $D = 4$  supergravity [55] which is invariant under local supersymmetry transformations. It is amazing that supergravity was not discovered much earlier than the 70s by people studied the massless spin-3/2 field, using the above reasoning for the construction of a consistent gravity – spin-3/2 field interacting system. In this respect let us cite what Fierz and Pauli ([10], page 226) wrote about the massless spin 3/2 field: “Whereas the theory for the spin value 2 has an important generalization for force fields, namely the gravitational theory, we here [in the case of spin 3/2] have no such a connection with a known theory. To get a generalization of the theory with interactions one would first of all have to find a physical interpretation of the gauge group, and the conservation theorem connected with this group”.

#### 4.2 Gravitational interaction of a spin $\frac{5}{2}$ field

Let us now, by analogy with supergravity, try to couple to gravity in four dimensions a field of spin  $\frac{5}{2}$  [17] which is described by the spin–tensor field  $\psi_{m_1 m_2}^\alpha$ . Again, the general coordinate invariance of the gravitational interactions requires that in the free field equations partial derivatives get replaced with covariant derivatives and the vector indices are contracted with the gravitational metric  $g_{mn}(x) = \eta_{mn} + \phi_{mn}(x)$ , where  $\phi_{mn}(x)$  is the deviation of the metric from the flat background which at the moment, for simplicity, we consider to be small and satisfy free spin 2 equations of motion. Thus the straightforward generalization of the equations of motion of the spin  $\frac{5}{2}$  field which describes its “minimal” interaction with gravity is

$$G_{\alpha m_1 m_2} = i\gamma_{\alpha\beta}^n (D_n \psi_{m_1 m_2}^\beta - D_{m_1} \psi_{nm_2}^\beta - D_{m_2} \psi_{nm_1}^\beta) = 0. \quad (4.8)$$

In (4.8) we have restored the imaginary unit  $i$  for further comparison with the massive Dirac equation  $(i\cancel{D} - m)\psi = 0$ .

We should now check whether these equations are invariant under the covariant modification of the higher spin gauge transformations

$$\delta\psi_{m_1 m_2}^\alpha(x) = D_{m_1} \xi_{m_2}^\alpha + D_{m_2} \xi_{m_1}^\alpha. \quad (4.9)$$

The gauge variation of the equations of motion (4.8) is [44]

$$\delta G_{m_1 m_2} = i[R_{m_1 n} \gamma^n \xi_{m_2} + R_{m_2 n} \gamma^n \xi_{m_1} - (R_{nm_1 m_2}^p + R_{nm_2 m_1}^p) \gamma^n \xi_p], \quad (4.10)$$

where I have suppressed the spinor index. We see that in a general gravitational background the variation does not vanish because of the presence of the Ricci tensor in the first two terms and of the Riemann tensor in the last term. If the bare Riemann tensor did not appear the spin  $\frac{5}{2}$  field equations would at least admit interactions with gravitational fields satisfying the free Einstein equations  $R_{mn} = 0$ . As we have discussed, this is the case for the gravitino field of spin  $\frac{3}{2}$  whose local supersymmetry transformations produce only the Ricci tensor term in the variation of the Rarita–Schwinger equations. Introducing an

appropriate supersymmetry variation of the graviton, one insures that the variation of the Rarita–Schwinger equations is proportional to the Einstein equations with the r.h.s. to be the energy–momentum tensor of the gravitino field. However, in the case of the higher spin fields with  $s \geq \frac{5}{2}$  the bare Riemann tensor always appears (as part of the Weyl tensor) in the variation of the field equations, and no way has been found to cancel such terms by adding non–minimal interaction terms and/or modifying the higher spin symmetry transformations (including that of the graviton), when the zero limit of the gravitation field corresponds to the flat Minkowski space (i.e. when the cosmological constant is zero).

So the conclusion has been made that in a space–time with zero cosmological constant it is not possible to construct a consistent gauge theory of interacting higher spin fields, which is in agreement with the general theorem of the possible symmetries of the S–matrix. But, as happens with many no–go theorems, sooner or later people find a way to circumvent them. In the case of the higher spins the way out has been found in constructing the theory in the AdS space, which has a non–zero cosmological constant  $\Lambda$ .

In the bosonic case Fronsdal and in the fermionic case Fang and Fronsdal [43] have generalized the free higher–spin field equations and actions to the AdS background. For instance, the equation of motion of the spin  $\frac{5}{2}$  field takes the following form

$$G_{m_1 m_2}^\alpha = i\gamma^{n\alpha}{}_\beta (\nabla_n \psi_{m_1 m_2}^\beta - \nabla_{m_1} \psi_{n m_2}^\beta - \nabla_{m_2} \psi_{n m_1}^\beta) - 2\Lambda^{\frac{1}{2}} \psi_{m_1 m_2}^\alpha = 0. \quad (4.11)$$

One can notice that the last term in (4.11) resembles a mass term of the spin  $\frac{5}{2}$  field, however the field has the number of physical degrees of freedom equal to that of the corresponding massless field in flat space, i.e. two states with helicities  $\pm\frac{5}{2}$ . This is because the equation of motion is invariant under the following gauge transformations

$$\delta\psi_{m_1 m_2}^\alpha(x) = \nabla_{m_1} \xi_{m_2}^\alpha + \nabla_{m_2} \xi_{m_1}^\alpha. \quad (4.12)$$

In (4.11) and (4.12)  $\nabla_m = D_m + \frac{i\Lambda^{\frac{1}{2}}}{2}\gamma_m$  is a so called  $SO(2,3)$  covariant derivative, and  $D_m$  is the standard covariant derivative in the AdS space whose Riemann curvature has the well known form

$$R_{mnp}^{(AdS) \ q} = -\Lambda(\delta_m^q g_{np} - \delta_n^q g_{mp}), \quad (4.13)$$

Remember also that the AdS metric is conformally flat  $g_{mn}^{AdS} = (1 - \Lambda x^p x_p)^{-2} \eta_{mn}$ . Note that the gamma–matrix  $\gamma_n$  entering (4.11) carries a curved vector index, it is hence non–constant and is related to the constant Dirac matrix  $\gamma_a$  (carrying a local Lorentz index) via the vielbein  $e_m^a(x)$  of the AdS space  $\gamma_n = e_m^a(x)\gamma_a$ .

It is important to notice that when acting on a spinor field  $\psi^\alpha(x)$  the commutator of  $\nabla_m$  is zero

$$[\nabla_m, \nabla_n]\psi^\alpha = 0 \quad (4.14)$$

while acting on a vector field  $V_p$  the commutator is

$$[\nabla_m, \nabla_n]V_p = [D_m, D_n]V_p = -R_{mnp}^{(AdS) \ q} V_q. \quad (4.15)$$

For the spin–tensor fields  $\psi_{m_1 \dots m_{s-\frac{1}{2}}}^\alpha$  we thus have

$$[\nabla_n, \nabla_p]\psi_{m_1 \dots m_{s-\frac{1}{2}}}^\alpha = [D_m, D_n]\psi_{m_1 \dots m_{s-\frac{1}{2}}}^\alpha = -\sum R_{mnm_1}^{(AdS) \ q} \psi_{q m_2 \dots m_{s-\frac{1}{2}}}^\alpha. \quad (4.16)$$

Note also that  $\nabla_m$  does not annihilate the gamma matrix  $\gamma_n$

$$\nabla_m \gamma_n = \frac{i\Lambda^{\frac{1}{2}}}{2} [\gamma_m, \gamma_n]. \quad (4.17)$$

Thus the gauge invariant field equations for higher spin fields in AdS do exist. If we now consider fluctuations of the gravitational field around the AdS background the gauge variation (4.10) of the equations (4.12) will again have contributions similar to (4.11) of the bare Riemann curvature of the fluctuating gravitational field

$$\delta G_{m_1 m_2} = -i(R_{nm_1 m_2}{}^p + R_{nm_2 m_1}{}^p) \gamma^n \xi_p + \dots, \quad (4.18)$$

where  $\dots$  stand for harmless terms, which can be canceled by an appropriate modification of the gauge transformations. As has been first noticed by Fradkin and Vasiliev in 1987 [24], because of the non-zero *dimensionful* cosmological constant of the background, it is now possible to modify the field equation (4.11) such that the variation of an appropriate additional term will cancel the dangerous Riemann curvature term in (4.18), at least in the first order of the perturbation of the gravitational field. For the spin  $\frac{5}{2}$  field the appropriate term describing its non-minimal coupling to a gravitational fluctuation is

$$\Delta G_{m_1 m_2} = \frac{i}{2\Lambda} (R_{pm_1 m_2 q} + R_{pm_2 m_1 q}) \nabla^q \psi^{pq}, \quad (4.19)$$

where  $R_{pm_1 m_2 q}$  is the Riemann curvature corresponding to the deviation of the gravitational field from the AdS background. Such a term can be obtained from a cubic interaction term in the spin  $\frac{5}{2}$  action

$$S_{int} = \frac{i}{\Lambda} \int d^D x \sqrt{g} \{ \bar{\psi}^{m_1 m_2} R_{pm_1 m_2 q} \nabla^q \psi^{pq} + \bar{\psi}^{m_1 m_2} (\nabla_r \gamma^r R_{pm_1 m_2 q}) \psi^{pq} \}. \quad (4.20)$$

Note that the cosmological constant enters the interacting term (4.19), (4.20) in a non-polynomial way. Therefore, such terms become singular when the cosmological constant tends to zero, and, does not allow of the flat space limit <sup>5</sup>.

Consider now the gauge variation of the interaction term (4.19)

$$\begin{aligned} \delta(\Delta G_{m_1 m_2}) &= \frac{2i}{\Lambda} (R_{pm_1 m_2 q} + R_{pm_2 m_1 q}) \gamma_n \nabla^n \nabla^p \xi^q \\ &= \frac{i}{\Lambda} (R_{pm_1 m_2 q} + R_{pm_2 m_1 q}) \gamma_n [\nabla^n, \nabla^p] \xi^q + \dots \end{aligned} \quad (4.21)$$

where  $\dots$  stand for the terms with the anticommutator of  $\nabla_n$  which are assumed to be harmless, i.e. can be canceled by an appropriate modification of the gauge transformations of fields and/or by adding more cubic interaction terms (with higher derivatives) in to the action and into the equation of motion. If we restrict ourselves to the consideration of only the first order in small gravitational interactions, then in (4.21) the commutator of derivatives should be restricted to the zero order contribution of the AdS curvature (4.13), (4.15). The AdS curvature (4.13) is proportional to the cosmological constant which cancels that in the denominator of (4.21). So in this approximation the form of the variation of the non-minimal interaction term (4.10) reduces to that of the rest of the field equation (4.18) with the opposite sign and thus cancels the latter.

<sup>5</sup>In the case of massive higher spin fields the mass plays the role of the dimensionful constant which, similar to  $\Lambda$ , can be used to construct electromagnetic and gravitational interactions of the massive higher spin fields even in flat space, however, such models usually suffer causality and unitarity problems

### 4.3 Towards a complete non-linear higher spin field theory

It turns out that beyond the linear approximation of gravitational fluctuations the situation with gauge invariance becomes much more complicated. As the analysis carried out by different people showed [20, 21, 24, 25], a gauge invariant interacting theory of massless higher spin fields should contain

- infinite number of fields of increasing spins involved in the interaction and in symmetry transformations and
- terms with higher derivatives of fields both in the action and in gauge transformations.

No complete action has been constructed so far to describe such an interacting theory of infinite number of higher spin fields, though generic non-linear equations of motion describing higher spin interactions do exist [26, 2]. A main problem is in finding and understanding the (non-abelian) algebraic structure of the gauge transformations of the higher spin fields modified by their interactions. In other words, the question is what is the gauge symmetry algebra which governs the interacting higher spin theory? Note that in the case of Yang–Mills vector fields and gravity the knowledge of the structure of non-abelian gauge symmetries and general coordinate invariance was crucial for the construction of the complete non-linear actions for these fields.

To deal simultaneously with the whole infinite tower of higher spins and to analyze their symmetry and geometrical properties, one may try to cast them into a finite number of ‘hyperfields’ by extending space–time with additional directions associated with infinitely many spin degrees of freedom. In the formulation discussed above this can be done by introducing auxiliary vector coordinates  $y^m$  [41].

Consider in a  $(D + D)$ -dimensional space parametrized by  $x^m$  and  $y^n$  a scalar field  $\Phi(x, y)$  which is analytic in  $y^n$ . Then  $\Phi(x, y)$  can be presented as a series expansion in powers of  $y^n$

$$\Phi(x, y) = \phi(x) + \phi_m(x) y^m + \phi_{m_1 m_2}(x) y^{m_1} y^{m_2} + \sum_{s=3}^{\infty} \phi_{m_1 \dots m_s}(x) y^{m_1} \dots y^{m_s}. \quad (4.22)$$

We see that the components of this expansion are D-dimensional symmetric tensor fields  $\phi_{m_1 \dots m_s}(x)$ . We would like  $\Phi(x, y)$  to have symmetry properties and to satisfy field equations which would produce the gauge transformations (2.1), the traceless conditions (2.5), (2.11) and the equations of motion (2.2) of the higher spin fields  $\phi_{m_1 \dots m_s}(x)$ . It is not hard to check that the gauge transformation of the hyperfield  $\Phi(x, y)$  should have the form

$$\delta\Phi(x, y) = y^m \partial_m \Xi(x, y), \quad (4.23)$$

where as above  $\partial_m = \frac{\partial}{\partial x^m}$ , and higher components of the gauge parameter  $\Xi(x, y) = \sum_{s=0}^{\infty} \xi_{m_1 \dots m_s}(x) y^{m_1} \dots y^{m_s}$  are traceless, which is ensured by imposing the condition

$$\partial_y^2 \Xi(x, y) = 0, \quad \partial_y^2 \equiv \eta^{mn} \frac{\partial}{\partial y^m} \frac{\partial}{\partial y^n} \quad (4.24)$$

The double tracelessness (2.11) is encoded in the equation

$$\partial_y^2 \partial_y^2 \Phi(x, y) = 0, \quad (4.25)$$

and the higher spin field equations (2.2) are derived from the following equations of motion of the hyperfield  $\Phi(x, y)$

$$\left[ \eta^{mn} - y^m \frac{\partial}{\partial y_n} + y^m y^n \partial_y^2 \right] \partial_m \partial_n \Phi(x, y) = 0. \quad (4.26)$$

Analogously, to describe the fields with half integer spins let us introduce a spinorial hyperfield

$$\Psi^\alpha(x, y) = \psi^\alpha(x) + \psi_m^\alpha(x) y^m + \sum_{s=\frac{5}{2}}^{\infty} \psi_{m_1 \dots m_{s-\frac{1}{2}}}^\alpha(x) y^{m_1} \dots y^{m_{s-\frac{1}{2}}}. \quad (4.27)$$

The gauge transformations of  $\Psi^\alpha(x, y)$  are

$$\delta \Psi^\alpha(x, y) = y^m \partial_m \Xi^\alpha(x, y), \quad \gamma^m \frac{\partial}{\partial y^m} \Xi^\alpha(x, y) = 0. \quad (4.28)$$

$\Psi^\alpha(x, y)$  satisfies the ‘triple’ gamma–traceless condition

$$\gamma^m \frac{\partial}{\partial y^m} \partial_y^2 \Psi(x, y) = 0, \quad (4.29)$$

and the equations of motion

$$\gamma^m \left[ \partial_m - y^n \frac{\partial}{\partial y^n} \partial_n \right] \Psi(x, y) = 0. \quad (4.30)$$

The equations (4.28), (4.29) and (4.30) comprise those for the half–integer spin fields.

The above construction is a simple example of how one can formulate the free theory of infinite number of higher spin fields in terms of a finite number of fields propagating in extended space. In contrast to the Fronsdal formulation, this construction is on the mass shell. It is not clear how to construct an action in the extended space which would produce the equations (4.26) and (4.30), neither how to generalize these equations to include non–linear terms.

Much more sophisticated on–shell formulations which involve either vector or spinor auxiliary variables and are based on a solid group–theoretical ground have been developed in [26, 27].

For instance, to find and study the algebraic and geometrical structure of higher spin symmetries (at least in  $D = 4$  and  $D = 6$ ), an alternative description of the higher spin fields has proved to be useful. It has been mainly developed by Vasiliev with collaborators (see [26], [2] and references therein) and by Sezgin and Sundell [5]. This is a so called unfolded formulation of the equations of motion of higher spin field theory with the use of spin–tensor representations of the Lorentz group and auxiliary commuting spinor coordinates. “Unfolded” basically means that all fields (including scalars and spinors) enter into

the game with their descendants, i.e. auxiliary fields which on the mass shell are higher derivatives of the physical fields. From the algebraic point of view the unfolded formulation is a particular realization and an extension of a so called free differential algebra which is also a basis of the group manifold approach [56]. The field equations are formulated as a zero curvature condition which requires also 0-forms to be involved into the description of systems with infinite number of degrees of freedom.

#### 4.4 Unfolded field dynamics

In the unfolded formulation [2] the fields of spin  $s \geq 1$  in  $D = 4$  are described by a generalized vielbein and connection one-form

$$\omega(x, Y) = \sum_{n,p=0}^{\infty} dx^m \omega_m^{A_1 \dots A_n, \dot{A}_1 \dots \dot{A}_p}(x) y_{A_1} \dots y_{A_p} \bar{y}_{\dot{A}_1} \dots \bar{y}_{\dot{A}_p} \quad (4.31)$$

and by its curvature two-form

$$R(x, Y) = d\omega(x, Y) - (\omega \wedge \star \omega)(x, Y), \quad (4.32)$$

where  $(y_A, \bar{y}_{\dot{A}}) = Y_{\alpha}$  ( $A, \dot{A} = 1, 2$ ) are auxiliary two-component Weyl spinor variables with even Grassmann parity which resemble twistors and satisfy the oscillator (or Moyal star-product) commutation relations

$$y_A \star y_B - y_B \star y_A = \epsilon_{AB}, \quad \bar{y}_{\dot{A}} \star \bar{y}_{\dot{B}} - \bar{y}_{\dot{B}} \star \bar{y}_{\dot{A}} = \epsilon_{\dot{A}\dot{B}}, \quad (4.33)$$

and the star-product of the connection has been used in the definition of the curvature (4.32).

Another object of the unfolded formulation is the zero-form

$$C(x, Y) = \sum_{n,p=0} C^{A_1 \dots A_n, \dot{A}_1 \dots \dot{A}_p}(x) y_{A_1} \dots y_{A_n} \bar{y}_{\dot{A}_1} \dots \bar{y}_{\dot{A}_p} \quad (4.34)$$

which contains the scalar field  $\phi(x) = C(x, Y)|_{Y=0}$ , the spinor field  $\psi^{\alpha}(x) = \frac{\partial}{\partial Y_{\alpha}} C(x, Y)|_{Y=0}$  and (Weyl) curvature tensors of the higher spin fields. The field  $C(x, Y)$  is introduced to incorporate the spin-0 and spin- $\frac{1}{2}$  matter fields, and its higher components in the  $Y$ -series expansion are either gauge field curvature tensors related to (4.32) or higher derivatives of the matter fields and of the gauge field curvatures.

The higher spin gauge transformations are

$$\delta \omega(x, Y) = d\xi(x, Y) - (\omega \star \xi)(x, Y) + (\xi \star \omega)(x, Y) \quad (4.35)$$

$$\delta C(x, Y) = (\xi \star C)(x, Y) - (C \star \xi)(x, Y), \quad (4.36)$$

where  $\tilde{\xi} = \xi(x, y, -\bar{y})$ .

In the free (linearized) higher spin theory, the following relation holds

$$\begin{aligned} R_{linear}(x, Y) = & \left\{ \partial^A \bar{\partial}^{\dot{B}} \omega(x, Y) \wedge \partial_A \bar{\partial}^{\dot{C}} \omega(x, Y) \right\} |_{Y=0} \bar{\partial}_{\dot{B}} \bar{\partial}_{\dot{C}} C(x, 0, \bar{y}) \\ & + \left\{ \partial^A \bar{\partial}^{\dot{B}} \omega(x, Y) \wedge \partial^C \bar{\partial}_{\dot{B}} \omega(x, Y) \right\} |_{Y=0} \partial_A \partial_C C(x, y, 0), \end{aligned} \quad (4.37)$$

and  $C(x, Y)$  satisfies the unfolded field equations

$$i\sigma_{A\dot{A}}^m \frac{\partial}{\partial x^m} C(x, Y) = \frac{\partial}{\partial y^A} \frac{\partial}{\partial \bar{y}^{\dot{A}}} C(x, Y), \quad (4.38)$$

which when written in components are equivalent to the free higher spin field equations considered in the symmetric tensor formulation. For details on the non-linear generalization of the unfolded equations (4.37) and (4.38) we refer the reader to [2] and references therein.

An advantage of the unfolded formalism is that it allows one to treat the whole infinite tower of the higher spin fields simultaneously and provides a compact form of the higher spin symmetry transformations which form an infinite dimensional associative Lie (super)algebra. All this is required, as we have discussed above, for the construction of the consistent interacting higher spin theory. The action describing the unfolded dynamics is still to be found though.

## 5. Other developments

### 5.1 Higher spin field theory from dynamics in tensorial spaces. An alternative to Kaluza–Klein.

The experience of studying various field theories teaches us that in many cases a new insight into their structure can be gained by finding and analyzing a classical dynamical object whose quantization would reproduce the field theory of interest. The well known examples are various spinning particle and superparticle models whose quantum dynamics is described by a corresponding (supersymmetric) field theory. In all the conventional cases only finite number of states of different spins can be produced by quantizing particle models. But, as we have mentioned, for a consistent interacting higher spin field theory we need an infinite number of states. These are produced by strings, but as it has been noted, the associated string field theory is rather complicated. It seems desirable at first to find and analyze a simpler model with an infinite number of quantum higher spin states.

Such a superparticle model does exist [58]. In addition to the relation to higher spins, this model reveals other interesting features, such as the invariance under supersymmetry with tensorial charges (which are usually associated with brane solutions of Superstring and M-Theory), and it has been the first example of a dynamical BPS system which preserves more than one half supersymmetry of the bulk. The study of these features was a main motivation for the original paper [58]. BPS states preserving  $\frac{2n-1}{2n}$  supersymmetries (with  $n = 16$  for  $D = 10, 11$ ) have later on been shown to be building blocks of any BPS state and conjectured to be hypothetical constituents or ‘preons’ of M-theory [59]. The relation of the model of [58] to the theory of massless higher spin fields in the unfolded formulation (4.38) was assumed in [60], where the quantum states of the superparticle was shown to form an infinite tower of the massless higher spin fields. This relation has been analyzed in detail in [62, 63, 64].

Probably, the first person who suggested a physical application of tensorial spaces to the theory of higher spins was C. Fronsdal.



In his Essay of 1985 [57] Fronsdal conjectured that four-dimensional higher spin field theory can be realized as a field theory on a ten-dimensional tensorial manifold parametrized by the coordinates

$$x^{\alpha\beta} = x^{\beta\alpha} = \frac{1}{2}x^m\gamma_m^{\alpha\beta} + \frac{1}{4}y^{mn}\gamma_{mn}^{\alpha\beta}, \quad m, n = 0, 1, 2, 3; \quad \alpha, \beta = 1, 2, 3, 4, \quad (5.1)$$

where  $x^m$  are associated with four coordinates of the conventional  $D = 4$  space-time and six tensorial coordinates  $y^{mn} = -y^{nm}$  describe spinning degrees of freedom.

The assumption was that by analogy with, for example,  $D = 10$  or  $D = 11$  supergravities, which are relatively simple theories but whose dimensional reduction to four dimensions produces very complicated extended supergravities, there may exist a theory in ten-dimensional tensorial space whose alternative Kaluza-Klein reduction may lead in  $D = 4$  to an infinite tower of fields with increasing spins instead of the infinite tower of Kaluza-Klein particles of increasing mass. The assertion was based on the argument that the symmetry group of the theory should be  $OSp(1|8) \supset SU(2, 2)$ , which contains the  $D = 4$  conformal group as a subgroup such that an irreducible (oscillator) representation of  $OSp(1|8)$  contains each and every massless higher spin representation of  $SU(2, 2)$  only once. So the idea was that using a single representation of  $OSp(1|8)$  in the ten-dimensional tensorial space one could describe an infinite tower of higher spin fields in  $D = 4$  space-time in a simpler way. Fronsdal regarded the tensorial space as a space on which  $Sp(8)$  acts like a group of generalized conformal transformations. Ten is the minimal dimension of such a space which can contain  $D=4$  space-time as a subspace. For some reason Fronsdal gave only a general definition and did not identify this ten-dimensional space with any conventional manifolds, like the ones mentioned above.

In his Essay Fronsdal also stressed the importance of  $OSp(1|2n)$  supergroups for the description of theories with superconformal symmetry. In the same period and later on different people studied  $OSp(1|2n)$  supergroups in various physical contexts. For instance,  $OSp(1|32)$  and  $OSp(1|64)$  have been assumed to be underlying superconformal symmetries of string- and M-theory.

The tensorial superparticle model of Bandos and Lukierski [58] turned out to be the first dynamical realization of the Fronsdal proposal.

The tensorial particle action has the following form

$$S[X, \lambda] = \int E^{\alpha\beta}(X(\tau)) \lambda_\alpha(\tau) \lambda_\beta(\tau), \quad (5.2)$$

where  $\lambda_\alpha(\tau)$  is an auxiliary commuting real spinor, a *twistor-like* variable, and  $E^{\alpha\beta}(x(\tau))$  is the pull back on the particle worldline of the tensorial space vielbein. In flat tensorial space

$$E^{\alpha\beta}(X(\tau)) = d\tau \partial_\tau X^{\alpha\beta}(\tau) = dX^{\alpha\beta}(\tau). \quad (5.3)$$

The dynamics of particles on the supergroup manifolds  $OSp(N|n)$  (which are the tensorial extensions of AdS superspaces) was considered for  $N = 1$  in [61, 64] and for a generic  $N$  in [62, 63]. The twistor-like superparticle in  $n = 32$  tensorial superspace was considered

in [66] as a point-like model for BPS preons [59], the hypothetical  $\frac{31}{32}$ -supersymmetric constituents of M-theory.

The action (5.2) is manifestly invariant under global  $GL(n)$  transformations. Without going into details which the reader may find in [58, 62, 64], let us note that the action (5.2) is invariant under global  $Sp(2n)$  transformations, acting non-linearly on  $X^{\alpha\beta}$  and on  $\lambda_\alpha$ , *i.e.* it possesses the symmetry considered by Fronsdal to be an underlying symmetry of higher spin field theory in the case  $n = 4$ ,  $D = 4$  [57].

Applying the Hamiltonian analysis to the particle model described by (5.2) and (5.3), one finds that the momentum conjugate to  $X^{\alpha\beta}$  is related to the twistor-like variable  $\lambda_\alpha$  via the constraint

$$P_{\alpha\beta} = \lambda_\alpha \lambda_\beta. \quad (5.4)$$

This expression is the direct analog and generalization of the Cartan–Penrose (twistor) relation for the particle momentum  $P_m = \lambda \gamma_m \lambda$ . In virtue of the Fierz identity  $\gamma_{m(\alpha\beta} \gamma_{\gamma)}^m = 0$  held in  $D = 3, 4, 6$  and 10 space-time, the twistor particle momentum is light-like in these dimensions. Therefore, in the tensorial spaces corresponding to these dimensions of space-time the first-quantized particles are massless [58, 60].

The quantum counterpart of (5.4) is the equation [60]

$$D_{\alpha\beta} \Phi(X, \lambda) = \left( \frac{\partial}{\partial X^{\alpha\beta}} - i \lambda_\alpha \lambda_\beta \right) \Phi(X, \lambda) = 0, \quad (5.5)$$

where the wave function  $\Phi(X, \lambda)$  depends on  $X^{\alpha\beta}$  and  $\lambda_\alpha$ . The general solution of (5.5) is the plane wave

$$\Phi(X, \lambda) = e^{iX^{\alpha\beta} \lambda_\alpha \lambda_\beta} \varphi(\lambda), \quad (5.6)$$

where  $\varphi(\lambda)$  is a generic function of  $\lambda_\alpha$ .

One can now Fourier transform the function (5.6) to another representation

$$C(X, Y) = \int d^4 \lambda e^{-iY^\alpha \lambda_\alpha} \Phi(X, \lambda) = \int d^4 \lambda e^{-iY^\alpha \lambda_\alpha + iX^{\alpha\beta} \lambda_\alpha \lambda_\beta} \varphi(\lambda). \quad (5.7)$$

The wave function  $C(X, Y)$  satisfies the Fourier transformed equation

$$\left( \frac{\partial}{\partial X^{\alpha\beta}} + i \frac{\partial^2}{\partial Y^\alpha \partial Y^\beta} \right) C(x, Y) = 0, \quad (5.8)$$

which is similar to the unfolded equation (4.38) and which actually reduces to the latter [62, 64].

Quantum states of the tensorial superparticle satisfying eq. (5.5) was shown to form an infinite series of massless higher spin states in  $D = 4, 6$  and 10 space-time [60]. In [61] quantum superparticle dynamics on  $OSp(1|4)$  was assumed to describe higher spin field theory in  $N = 1$  super  $AdS_4$ .

In [60] it was shown explicitly how the alternative Kaluza–Klein compactification produces higher spin fields. It turns out that in the tensorial superparticle model, in contrast to the conventional Kaluza–Klein theory, the compactification occurs in the momentum

space and not in the coordinate space. The coordinates conjugate to the compactified momenta take discrete (integer and half integer values) and describe spin degrees of freedom of the quantized states of the superparticle in conventional space-time.

In [62] M. Vasiliev has extensively developed this subject by having shown that the first-quantized field equations (5.8) in tensorial superspace of a bosonic dimension  $\frac{n(n+1)}{2}$  and of a fermionic dimension  $nN$  are  $OSp(N|2n)$  invariant, and for  $n = 4$  correspond to the unfolded higher spin field equations in  $D = 4$ . It has also been shown [63] that the theory possesses properties of causality and locality.

As was realized in [62, 63], the field theory of quantum states of the tensorial particle is basically a classical theory of two fields in the tensorial space, a scalar field  $b(X^{\alpha\beta})$  and a spinor field  $f^\alpha(X^{\beta\gamma})$ . These fields form a fundamental linear representation of the group  $OSp(1|2n)$  and satisfy the following tensorial equations

$$(\partial_{\alpha\beta}\partial_{\gamma\delta} - \partial_{\alpha\gamma}\partial_{\beta\delta})b(X) = 0, \quad \partial_{\alpha\beta}f_\gamma(X) - \partial_{\alpha\gamma}f_\beta(X) = 0. \quad (5.9)$$

In the case of  $n = 4$  (5.1) the fields  $b(X)$  and  $f_\alpha(X)$  subject to eqs. (5.9) describe the infinite tower of the massless (conformally invariant) fields of all possible integer and half-integer spins in the physical four-dimensional subspace of the ten-dimensional tensorial space [57, 62]. In the cases of  $n = 8$  and  $n = 16$  which correspond to  $D = 6$  and  $D = 10$  space-time, respectively, the equations (5.9) describe conformally invariant higher spin fields with self-dual field strengths [53].

Let us consider in more detail the case of  $n = 4$  and  $D = 4$  we split  $X^{\alpha\beta}$  onto  $x^m$  and  $y^{mn}$  as in eq. (5.1), the system of equations (5.9) takes the form

$$\begin{aligned} \partial_p \partial^p b(x^l, y^{mn}) &= 0, \quad \partial_p \partial_q b(x^l, y^{mn}) - 4\partial_{pr} \partial^r_q b(x^l, y^{mn}) = 0, \quad \partial_q^p \partial_p b(x^l, y^{mn}) = 0, \\ \epsilon^{pqrt} \partial_q \partial_{rt} b(x^l, y^{mn}) &= 0, \quad \epsilon^{pqrt} \partial_{pq} \partial_{rt} b(x^l, y^{mn}) = 0, \end{aligned} \quad (5.10)$$

$$\gamma^p \partial_p f(x^l, y^{mn}) = 0, \quad [\partial_p - 2\gamma^r \partial_{rp}] f(x^l, y^{mn}) = 0, \quad (5.11)$$

where  $\partial_p$  and  $\partial_{rp}$  are the derivatives along  $x^p$  and  $y^{rp}$ , respectively.

Then let us expand  $b(x, y)$  and  $f_\alpha(x, y)$  in series of  $y^{mn}$

$$\begin{aligned} b(x^l, y^{mn}) &= \phi(x) + y^{m_1 n_1} F_{m_1 n_1}(x) + y^{m_1 n_1} y^{m_2 n_2} [R_{m_1 n_1, m_2 n_2}(x) - \frac{1}{2} \eta_{m_1 m_2} \partial_{n_1 n_2} \phi(x)] \\ &\quad + \sum_{s=3}^{\infty} y^{m_1 n_1} \dots y^{m_s n_s} [R_{m_1 n_1, \dots, m_s n_s}(x) + \dots], \end{aligned} \quad (5.12)$$

$$\begin{aligned} f^\alpha(x^l, y^{mn}) &= \psi^\alpha(x) + y^{m_1 n_1} [\mathcal{R}_{m_1 n_1}^\alpha(x) - \frac{1}{2} \partial_{m_1} (\gamma_{n_1} \psi)^\alpha] \\ &\quad + \sum_{s=\frac{5}{2}}^{\infty} y^{m_1 n_1} \dots y^{m_{s-\frac{1}{2}} n_{s-\frac{1}{2}}} [\mathcal{R}_{m_1 n_1, \dots, m_{s-\frac{1}{2}} n_{s-\frac{1}{2}}}^\alpha(x) + \dots]. \end{aligned}$$

In (5.12)  $\phi(x)$  and  $\psi^\alpha(x)$  are scalar and spin 1/2 field,  $F_{m_1 n_1}(x)$  is the Maxwell field strength,  $R_{m_1 n_1, m_2 n_2}(x)$  is the curvature tensor of the linearized gravity,  $\mathcal{R}_{m_1 n_1}^\alpha(x)$  is the Rarita-Schwinger field strength and other terms in the series stand for generalized Riemann curvatures of spin- $s$  fields (which also contain contributions of derivatives of lower spin fields denoted by dots, as in the case of the Rarita-Schwinger field and gravity). The scalar and the spinor field satisfy, respectively, the Klein-Gordon and the Dirac equation, and the

higher spin field curvatures satisfy the Bianchi identities (3.5), (3.6) and the linearized higher spin field equations (3.10) and (3.11) in  $D = 4$  space-time. Similar equations also follow from the unfolded equations (4.38). In the model under consideration they are consequences of the field equations (5.9), or equivalently of (5.10) and (5.11) in the flat tensorial space. The generalization of the equations (5.9) to a field theory on the tensorial manifold  $OSp(1|n)$ , which for  $n = 4$  corresponds to the theory of higher spin fields in  $AdS_4$ , has been derived in [64].

An interesting and important problem is to find a simple and appropriate non-linear generalization of equations (refbf which would correspond to an interacting higher spin field theory. An attempt to construct such a generalization in the framework of tensorial superspace supergravity was undertaken in [65].

## 5.2 Massless higher spin field theory as a tensionless limit of superstring theory

In these lectures we have considered the formulations of massless higher spin field theory which are not directly related to String Theory. A natural question arises which formulation one can derive from String Theory at the tensionless limit  $T \sim \frac{1}{\alpha'} \rightarrow 0$ . This has been a subject of a number of papers [67]–[70] (and references therein) which we briefly sketch below.

Consider, for instance a free open bosonic string in flat space-time, whose worldsheet is parametrized by a ‘spatial’ coordinate  $\sigma \in [0, \pi]$  and a ‘time’ coordinate  $\tau$ . String dynamics is described by the coordinates

$$X^m(\tau, \sigma) = x^m + 2\alpha' p^m \tau + i\sqrt{2\alpha'} \sum_{n \neq 0}^{\infty} \frac{1}{n} a_n^m e^{-in\tau} \cos(n\sigma) \quad (5.13)$$

and momenta

$$P^m(\tau, \sigma) = p^m + \frac{1}{\sqrt{2\alpha'}} \sum_{n \neq 0}^{\infty} a_n^m e^{-in\tau} \cos(n\sigma), \quad (5.14)$$

where  $x^m$  and  $p^m$  are the center of mass variables and  $a_n^m$  are the string oscillator modes satisfying (upon quantization) the commutation relations  $[p^m, x^p] = -i\eta^{mp}$ ,  $[a_n^m, a_l^p] = n\delta_{n+l}\eta^{mp}$ .

String dynamics is subject to the Virasoro constraints

$$L_k = \frac{1}{2} \sum_{n=-\infty}^{+\infty} a_{k-n}^m a_{m n} = \sqrt{2\alpha'} p^m a_{m k} + \frac{1}{2} \sum_{n \neq k, 0} a_{k-n}^m a_{m n}, \quad k \neq 0, \quad (5.15)$$

$$L_0 = 2\alpha' p^m p_m + \sum_{n>0} a_{-n}^m a_{m n}. \quad (5.16)$$

The latter produces the mass shell condition for the string states

$$M^2 = -p^m p_m = \frac{1}{2\alpha'} \sum_{n>0} a_{-n}^m a_{m n}. \quad (5.17)$$

We observe that in the tensionless limit  $\alpha' \rightarrow \infty$  all string states become massless, while the properly rescaled Virasoro constraints become at most linear in the oscillator modes

$$l_0 = \frac{1}{2\alpha'} L_0|_{\alpha' \rightarrow \infty} = p^m p_m, \quad l_k = \frac{1}{\sqrt{2\alpha'}} L_k|_{\alpha' \rightarrow \infty} = p_m a_k^m \quad (5.18)$$

and satisfy a simple algebra without any central charge

$$[l_0, l_k] = 0, \quad [l_j, l_k] = \delta_{j+k} l_0. \quad (5.19)$$

Thus in the tensionless limit the quantum consistency of string theory does not require any critical dimension for the string to live in. Note that at  $\alpha' \rightarrow \infty$  the string coordinate (5.13) blows up and is not well defined, while the oscillator modes remain appropriate variables for carrying out the quantization of the theory.

The corresponding nilpotent BRST charge takes the form

$$Q = \sum_{n=-\infty}^{+\infty} (c_{-n} l_n - \frac{n}{2} b_0 c_{-n} c_n), \quad (5.20)$$

where  $c_n$  and  $b_n$  are the ghosts and anti-ghosts associated with the constraint algebra (5.19).

The BRST charge can be used to construct a free action for the string field states  $|\Phi\rangle$ , obtained by acting on the Fock vacuum by the creating operators,

$$S = \frac{1}{2} \int \langle \Phi | Q | \Phi \rangle. \quad (5.21)$$

The action (5.21) can be used for the derivation of a corresponding action and equations of motions of the higher spin fields encoded in  $|\Phi\rangle$ . As has been shown in [70] such an action and equations of motion are more involved than eqs. (2.15), (2.16), (2.2) and (2.3) since they contain intertwined (triplet) fields of different spins. The equations (2.15), (2.16), (2.2) and (2.3) are obtained from (5.21) upon gauge fixing part of the available local symmetry and by eliminating auxiliary fields.

Further details the interested reader can find in [69, 70] and references therein. We should note that the tensionless limit of the string considered here differs from the so called null string models [71]. In these models, in contrast to the way of getting the tensionless string discussed above the limit is taken in such a way that the string coordinate  $X^m(\sigma, \tau)$  remains a well defined variable, while the oscillator modes disappear. As a result the quantum states of the null strings correspond to a continuous set of massless particles without (higher) spin.

## 6. Conclusion

In these lectures we have described main features and problems of higher spin field theory and have flashed some ways along which it has been developed over last years.

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